MAA OMWATI DEGREE COLLEGE
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EXAM NOTES – B.SC 4TH SEM

SUBJECT – REAL ANALYSIS

SUBJECT CODE-25 MATM404DS01

UNIT 1

FINITE AND FINITE SETS
It sounds like you're diving into the building blocks of set theory! The distinction between finite and infinite sets is pretty intuitive once you look at the "end" of the list.

Here is a breakdown of how they differ, how to identify them, and a few "brain-bender" examples.

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1. Finite Sets

A **finite set** is a set that has a specific number of elements. If you started counting the members of the set, you would eventually reach a point where you've counted them all and have to stop.

- **Definition:** A set A is finite if there exists a natural number n such that the number of elements in A is exactly n.
- **Cardinality:** The count of elements (denoted as $n(A)$ or $|A|$) is a whole number.

Examples:

- $A=\{2,4,6,8,10\}$ (Cardinality is 5)
- The set of days in a week.
- The set of all humans currently living on Earth (it's a massive number, but it is still a specific, countable total).

2. Infinite Sets

An **infinite set** is a set whose elements cannot be counted to a finish. No matter how long you count, there will always be more elements remaining.

- **Definition:** A set that is not finite. It continues without end.
- **Notation:** Usually represented with an ellipsis (...) at the end to show the pattern continues forever.

Examples:

- The set of all natural numbers: $N=\{1,2,3,4,...\}$
- The set of points on a line segment.
- The set of all even integers.

Key Differences at a Glance

Feature	Finite Set	Infinite Set
Countability	Ends at a specific number.	Never ends.
Cardinality	A natural number (e.g., 10, 500).	Represented by symbols like $\aleph_0$ (Aleph-null).
Example	$\{x \mid x \text{ is a vowel in English}\}$	$\{x \mid x \text{ is a prime number}\}$
Subsets	A proper subset always has fewer elements.	A proper subset can have the "same" number of elements (Paradoxical!).

A Quick Note on "Empty Sets"

Is an empty set $\emptyset$ or $\{\}$ finite or infinite? It is **finite**. Its cardinality is 0, which is a defined whole number.

The "Infinite" Mind-Blower

Not all infinite sets are the same size! For example, the set of all "real numbers" (including decimals) is actually "larger" than the set of "counting numbers," even though both are infinite. This is a concept known as **uncountable vs. countable infinity**.

COUNTABLE AND UNCOUNTABLE SETS
If finite vs. infinite was the introduction, **Countable vs. Uncountable** is where set theory gets really wild. This distinction deals with the different "sizes" (cardinalities) of infinity.

The core idea comes from Georg Cantor, who realized that some infinities are literally bigger than others.

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1. Countable Sets

A set is **countable** if you can list its elements one by one (even if the list goes on forever). If you can assign a "counting number" to every item in the set without skipping any, it is countable.

- **Finite Sets:** All finite sets are countable (e.g., the colors of a rainbow).
- **Countably Infinite Sets:** These are infinite sets that have the same "size" as the natural numbers ($N=\{1,2,3,...\}$).
- **The Symbol:** The cardinality of a countably infinite set is denoted as $\aleph_0$ (Aleph-null).

Surprising Examples:

- **Integers (Z):** Even though it includes negative numbers, you can count them by alternating: $0,1,-1,2,-2...$
- **Rational Numbers (Q):** Even though there are infinite fractions between 1 and 2, Cantor proved you can list them all using a grid-diagonal method.

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2. Uncountable Sets

An **uncountable set** is an infinite set that is so "dense" or "large" that it is impossible to create a list of its members. You will always miss some, no matter how clever your counting system is.

- **The Concept:** If you try to map these to natural numbers, you will always run out of natural numbers before you finish labeling the set.
- **The Symbol:** The cardinality of the real numbers is often denoted as c (the cardinality of the continuum) or $2^{\aleph_0}$.

Examples:

- **Real Numbers ($\mathbb{R}$):** This includes all decimals. Between 0 and 1, there are more numbers than there are integers in the entire universe.
- **Irrational Numbers:** Numbers like π or 2.
- **The Power Set of Natural Numbers:** The set of all possible subsets of $\{1,2,3,...\}$.

Summary Comparison

Feature	Countable Sets	Uncountable Sets
Can you list them?	Yes (e.g., $a_1, a_2, a_3...$)	No. Any list is incomplete.
Smallest Example	Natural Numbers ($\mathbb{N}$)	Real Numbers ($\mathbb{R}$)
Density	Can be "gaps" between items.	Usually "continuous" (no gaps).
Cardinality	n (finite) or $\aleph_0$ (infinite)	Greater than $\aleph_0$

How do we prove it? (Cantor's Diagonal Argument)

To prove that real numbers are uncountable, Cantor used a "proof by contradiction." He imagined a list of all decimals between 0 and 1. He then showed he could construct a new decimal by changing the 1^{st} digit of the 1^{st} number, the 2^{nd} digit of the 2^{nd} number, and so on.

This new number is guaranteed to be different from every single number on the "infinite" list. Therefore, the list was never complete!

BOUNDED SETS AND BOUNDED SETS

When we talk about **bounded sets**, we are essentially asking: *"Does this set stay within a specific 'fence,' or does it go off to infinity?"* In mathematical terms, a set is bounded if it has both a **lower limit** and an **upper limit**.

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1. Bounded Sets

A set S is **bounded** if there is a number that is smaller than everything in the set (Lower Bound) and a number that is larger than everything in the set (Upper Bound).

- **Bounded Above:** There exists a real number M such that $x \leq M$ for all x in S. (M is the upper bound).
- **Bounded Below:** There exists a real number m such that $x \geq m$ for all x in S. (m is the lower bound).
- **Bounded:** A set is simply "bounded" only if it is bounded **both** above and below.

Examples:

- **The interval $[0,10]$:** This is bounded. 0 is a lower bound and 10 is an upper bound.
- **Finite Sets:** Any finite set, like $\{2,5,9\}$, is always bounded.
- **Sine/Cosine Values:** The set of all values of $\sin(x)$ is bounded between -1 and 1.

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2. Unbounded Sets

A set is **unbounded** if it fails to have an upper bound, a lower bound, or both. These sets "escape" to infinity in at least one direction.

Examples:

- **Natural Numbers ($\mathbb{N}$):** $\{1,2,3,...\}$ is bounded **below** (by 1), but it is **unbounded above** because it goes to $+\infty$.
- **Integers ($\mathbb{Z}$):** $\{...,-2,-1,0,1,2,...\}$ is **unbounded** in both directions ($-\infty$ to $+\infty$).
- **The set $\{x : x > 5\}$:** Bounded below by 5, but unbounded above.

3. Supremum and Infimum

When discussing bounded sets, mathematicians use two specific terms to describe the "best" possible bounds:

1. **Supremum (Least Upper Bound):** The smallest possible number that is still greater than or equal to everything in the set.
2. **Infimum (Greatest Lower Bound):** The largest possible number that is still less than or equal to everything in the set.

Example: Consider the open interval $(0,1)$.

- The set doesn't actually contain 0 or 1.
- However, the **Infimum is 0** and the **Supremum is 1**. Even though they aren't "in" the set, they act as the tightest possible "fence" around it.

Summary Table

Set Type	Has Upper Bound?	Has Lower Bound?	Example
Bounded	Yes	Yes	$[0,1]$
Bounded Above Only	Yes	No	$\{x \in \mathbb{R} : x \leq 5\}$
Bounded Below Only	No	Yes	$\{1,2,3,4,...\}$
Unbounded	No	No	The set of all Real Numbers ($\mathbb{R}$)

SUPREMA AND INFIMA

Think of **Suprema** and **Infima** as the "tightest possible fences" around a set.

While a set can have many upper and lower bounds, the **Supremum** and **Infimum** are the most precise ones. They are the "limit" of where the set ends, even if the set doesn't actually contain that boundary point.

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1. The Supremum (Least Upper Bound)

The **Supremum** (abbreviated as $\sup$) of a set S is the smallest real number that is greater than or equal to every element in S.

- **Rule 1:** It must be an upper bound (nothing in the set is bigger).
- **Rule 2:** If you pick any number slightly smaller than the supremum, it is no longer an upper bound (at least one element of the set will jump over it).

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2. The Infimum (Greatest Lower Bound)

The **Infimum** (abbreviated as $\inf$) of a set S is the largest real number that is less than or equal to every element in S.

- **Rule 1:** It must be a lower bound (nothing in the set is smaller).
- **Rule 2:** If you pick any number slightly larger than the infimum, it is no longer a lower bound.

The Big Difference: Bound vs. Maximum/Minimum

A common point of confusion is whether the Supremum is the same as the "Maximum."

- **Maximum/Minimum:** Must be a member of the set.
- **Supremum/Infimum:** Does **not** have to be a member of the set.

Example: The Open Interval $(0,1)$

Consider the set of all numbers between 0 and 1, *excluding* 0 and 1 themselves.

- **Upper Bounds:**$1, 5, 100, \pi$.
- **Supremum (sup):**$\mathbf{1}$. (It is the *least* of all upper bounds).
- **Maximum:None**. (Because 1 is not in the set, and for any number like 0.99, you can always find 0.999).

- **Lower Bounds:**$0, -1, -50$.
- **Infimum ($\inf$):0**. (It is the *greatest* of all lower bounds).
- **Minimum:None**. (Because 0 is not in the set).

Comparison Table

Set	Infimum ($\inf$)	Minimum ($\min$)	Supremum ($\sup$)	Maximum ($\max$)
$\{1,2,3\}$	1	1	3	3
$[0,5]$	0	0	5	5
$(0,5)$	0	*None*	5	*None*
$\{\frac{1}{n} : n \in \mathbb{N}\}$	0	*None*	1	1

Note on the last example: For the set $\{1, 1/2, 1/3, 1/4, ...\}$, the numbers get closer and closer to 0 but never reach it. Therefore, 0 is the Infimum, but there is no Minimum element.

The Completeness Axiom

This is the "Golden Rule" of real numbers: Every non-empty set of real numbers that is bounded above **must** have a least upper bound (Supremum) that is also a real number. This is what makes the number line "solid" without any tiny holes.

COMPLETENESS PROPERTY OF R

The **Completeness Property of $\mathbb{R}$** (also known as the **Least Upper Bound Property**) is what separates the Real numbers from the Rational numbers.

Think of it as the "no holes" rule. It guarantees that the real number line is a solid, continuous string without any microscopic gaps.

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1. The Formal Definition

A set of numbers is **complete** if every non-empty subset that is bounded above has a **Supremum** (Least Upper Bound) that is also a member of that set.

In the case of Real Numbers ($\mathbb{R}$):

If a set $S \subset \mathbb{R}$ is non-empty and has an upper bound, then there exists a real number L such that $L = \sup(S)$.

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2. Why $\mathbb{Q}$ (Rational Numbers) is NOT Complete

To understand why $\mathbb{R}$ is special, look at the Rational numbers (fractions).

Imagine a set A of all rational numbers whose square is less than 2:

$$A = \{x \in \mathbb{Q} : x^2 < 2\}$$

- This set is bounded above (for example, by 2 or 1.5).
- As you get closer and closer to the "edge" of this set, you are approaching 2.
- **The Problem:**2 is **irrational**. It is not in $\mathbb{Q}$.

Because the "limit" or "boundary" of this set is a hole in the rational number line, we say $\mathbb{Q}$ lacks the Completeness Property. In the Real numbers ($\mathbb{R}$), that hole is filled by 2.

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3. Equivalent Ways to Describe Completeness

The Completeness Property is so fundamental that it can be described in several different ways. If any one of these is true, the others are usually true as well:

- **Monotone Convergence Theorem:** Every bounded, monotonic (always increasing or always decreasing) sequence of real numbers converges to a real number.
- **Bolzano-Weierstrass Theorem:** Every bounded sequence has a convergent subsequence.
- **Nested Interval Property:** If you have a sequence of closed intervals $[a_n, b_n]$ that keep getting smaller and are nested inside each other, there is at least one real number that is in **all** of them.
- **Cauchy Completeness:** Every Cauchy sequence (a sequence where the terms get closer and closer to each other) converges to a limit within the set.

4. Why Does This Matter?

Without the Completeness Property, most of Calculus would fall apart:

1. **Intermediate Value Theorem:** You couldn't guarantee that a continuous function crossing from negative to positive actually hits zero; it might just "jump" over the hole.
2. **Extreme Value Theorem:** You couldn't guarantee a function reaches a maximum or minimum on a closed interval.
3. **Limits:** Many limits that define derivatives and integrals wouldn't "exist" because the number they converge to wouldn't be in your system.

Summary Comparison

Feature	Rational Numbers ($\mathbb{Q}$)	Real Numbers ($\mathbb{R}$)
Gaps?	Yes (at every irrational point).	No (completely solid).
Bounded Sets	May not have a supremum in $\mathbb{Q}$.	Always have a supremum in $\mathbb{R}$.
Calculus?	Impossible.	Possible.

ARCHIMEDIAN PROPERTY OF R

The **Archimedean Property** is a fundamental principle of the real number system ($\mathbb{R}$) that essentially says: **"No matter how small a positive number is, if you add it to itself enough times, you can eventually surpass any other number."**

It bridges the gap between the incredibly small (infinitesimals) and the incredibly large (infinity), ensuring that there are no "infinitely large" or "infinitely small" real numbers.

1. The Formal Statement

There are a few ways to state this property, but the most common version is:

For any real number $x \in \mathbb{R}$, there exists a natural number $n \in \mathbb{N}$ such that $n > x$.

In other words, the set of natural numbers $\mathbb{N}=\{1,2,3,...\}$ is **not bounded above** in $\mathbb{R}$.

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2. Alternative Versions (The "Practical" Views)

Mathematically, these three statements are all equivalent:

1. **The "Giant" Version:** For any two positive real numbers a and b, you can find a natural number n such that $na>b$.
   - *Translation:* Even if you have a tiny spoon (a) and a giant bathtub (b), if you scoop enough times (n), you will eventually empty the tub.
2. **The "Tiny" Version:** For any $\epsilon>0$ (a tiny positive number), there exists a natural number n such that $\frac{1}{n}<\epsilon$.
   - *Translation:* You can make a fraction as small as you want just by making the denominator large enough.
3. **The "Integer" Version:** For any real number x, there is always an integer n such that $n-1\leq x<n$.

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3. Why is it Important?

The Archimedean Property is a direct consequence of the **Completeness Property** we discussed earlier. It is the reason we can perform many operations in Calculus and Analysis:

- **Proving Limits:** It allows us to prove that the sequence $\frac{1}{n}$ converges to 0. Without this property, there might be some "tiny" number that stays above zero forever, no matter how large n gets.
- **Density of Rationals:** It helps prove that between any two distinct real numbers, there is always a rational number (a fraction).
- **Measurement:** It ensures that our system of measurement works consistently. We can measure any length using a fixed unit by repeating that unit enough times.

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4. Does it apply to all number systems?

Surprisingly, **no**. While the Real numbers ($\mathbb{R}$) and Rational numbers ($\mathbb{Q}$) are Archimedean, there are "Non-Archimedean" fields used in advanced mathematics (like

p-adic numbers or hyperreal numbers) where you *can* have numbers that are infinitely large or infinitely small.

Summary Table

Property	Reality in $\mathbb{R}$
Is $\mathbb{N}$ bounded?	No. You can always find a larger integer.
Can $\frac{1}{n}$ be smaller than any ϵ?	Yes. Just pick a large enough n.
Is there a largest Real number?	No.
Is there a smallest positive Real number?	No. (0 is the infimum, but not a positive member).

For any real number $x \in \mathbb{R}$, there exists a natural number $n \in \mathbb{N}$ such that $n > x$.

In other words, the set of natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$ is **not bounded above** in $\mathbb{R}$.

Is there a smallest positive Real number?	No. (0 is the infimum, but not a positive member).

UNIT -2

RIEMANN INTEGRAL

The **Riemann Integral** is the formal definition of the "area under a curve" that you typically learn in introductory Calculus. Created by Bernhard Riemann, it uses the concept of **partitions** and **sums** to turn a continuous area into a calculation.

It relies heavily on the concepts we just discussed: **Suprema**, **Infima**, and **Bounded Sets**.

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1. How It Works: The Partition

To find the area under a function $f(x)$ on an interval $[a,b]$, we don't try to calculate the whole shape at once. Instead:

1. **Divide:** We chop the interval $[a,b]$ into smaller sub-intervals. This "chopping" is called a **Partition (P)**.
2. **Approximate:** In each small sub-interval, we draw a rectangle.
3. **Sum:** We add up the areas of all these rectangles.

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2. Upper and Lower Riemann Sums

Since a curve isn't flat, how do we choose the height of the rectangles? We use the "fences" we learned about earlier:

- **Lower Riemann Sum ($L(P,f)$):** We use the **Infimum** of the function in each sub-interval. This creates rectangles that stay *under* the curve.
- **Upper Riemann Sum ($U(P,f)$):** We use the **Supremum** of the function in each sub-interval. This creates rectangles that cover *more* than the curve.

As the partition gets "finer" (meaning the rectangles get thinner and thinner), the gap between the Upper Sum and the Lower Sum starts to shrink.

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3. The Definition of "Integrable"

A function is **Riemann Integrable** if, as the width of the rectangles approaches zero, the Upper Sum and the Lower Sum converge to the exact same number.

Mathematically, we define:

- **Lower Integral:** The supremum of all possible lower sums.
- **Upper Integral:** The infimum of all possible upper sums.

If the **Lower Integral = Upper Integral**, that common value is the **Riemann Integral**, denoted as:

$\int_a^b f(x)dx$

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4. Criteria for Integrability

Not every function can be integrated using Riemann's method. For a function to be Riemann integrable on $[a,b]$:

1. **It must be Bounded:** If the function goes to infinity (like $1/x$ at $x=0$), the rectangles would have infinite height, and the sum wouldn't work.
2. **It must be "Mostly" Continuous:** Continuous functions are always integrable. Functions with a few "jumps" (discontinuities) are also usually fine.

The "Classic" Fail: The **Dirichlet Function**.

$$f(x)=\begin{cases}1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational}\end{cases}$$

Because of the **Density of Rationals**, every tiny interval contains both 1s and 0s. The Upper Sum will always be 1, and the Lower Sum will always be 0. They never meet, so this function is **not** Riemann integrable!

Summary Table

Concept	Role in Integration
Partition	Breaking the x-axis into segments.
Infimum	Determines the height of the "Lower" rectangle.
Supremum	Determines the height of the "Upper" rectangle.
Limit	Making the rectangles infinitely thin to find the exact area .

INTEGRABILITY OF CONTINUOUS AND MONOTONIC FUNCTION

We've established that for a function to be **Riemann Integrable**, the "Gap" between the Upper Sum (U) and the Lower Sum (L) must be able to be made as small as we want.

Two specific types of functions—**Continuous** and **Monotonic**—are guaranteed to be integrable. Here is the mathematical "why" behind both.

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1. Integrability of Continuous Functions

If a function $f(x)$ is continuous on a closed interval $[a,b]$, it is **always** Riemann integrable.

The Logic:

Continuous functions are well-behaved. On a closed interval, a continuous function is **Uniformly Continuous**. This means that if you make the width of your rectangles (the partition) small enough, the difference between the highest point (Supremum) and the lowest point (Infimum) in *every* sub-interval becomes tiny.

- Because the function doesn't "jump" or "spike," the Upper Rectangles and Lower Rectangles eventually hug the curve so tightly that their areas become identical in the limit.

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2. Integrability of Monotonic Functions

A **Monotonic** function is one that is either entirely non-increasing or entirely non-decreasing. Surprisingly, a monotonic function is **always** integrable, even if it has many "jump" discontinuities!

The Logic:

Let's assume $f(x)$ is increasing on $[a,b]$.

1. In any sub-interval $[x_{i-1},x_i]$, the **Infimum** is always at the left endpoint, and the **Supremum** is always at the right endpoint.
2. When you calculate $U(P,f)-L(P,f)$, the terms in the middle of the sum "telescope" (cancel each other out).
3. The total difference between the Upper and Lower sums simplifies to:

$$(U-L)=\text{Width of rectangles}\times[f(b)-f(a)]$$

As long as the width of your rectangles goes to zero, the difference between the sums goes to zero. It doesn't matter if the function "jumps" as it goes up; because it's monotonic, those jumps are controlled and finite.

Comparison of Integrability

Function Type	Integrable?	Reason

Function Type	Integrable?	Reason
Continuous	**Yes**	Uniform continuity ensures the height difference $(\sup - \inf)$ in each rectangle vanishes.
Monotonic	**Yes**	The "telescoping" nature of the sums ensures the total error is bounded by the function's total rise.
Bounded with finite jumps	**Yes**	The area "under" a single point (the jump) is zero, so it doesn't affect the total integral.
Unbounded	**No**	If $f(x) \to \infty$, the Riemann sum cannot be defined.

The "Integrability Criterion" Summary

A bounded function f is integrable on $[a,b]$ if and only if for every $\epsilon > 0$, there exists a partition P such that:

$$U(P,f) - L(P,f) < \epsilon$$

- For **Continuous** functions, we control the "height" difference using ϵ.
- For **Monotonic** functions, we control the "width" of the partition to make the error disappear.

FUNDAMENTAL THEOREM OF INTEGRAL CALCULAS

The **Fundamental Theorem of Calculus (FTC)** is the "great bridge" of mathematics. It links the two main branches of calculus—**Differential Calculus** (slopes and rates of change) and **Integral Calculus** (areas and accumulation)—showing that they are inverse operations of each other.

The theorem is usually split into two parts.

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1. FTC Part 1: The Accumulation Function

This part tells us that if you define a function as the "area accumulated so far" under a curve, the derivative of that area function is the original curve itself.

The Statement: If f is continuous on $[a,b]$, and we define a new function $F(x)$ as:

$F(x)=\int_a^x f(t)\,dt$

Then:

$F'(x)=f(x)$

What this means: Integration and Differentiation are opposites. If you integrate a function and then differentiate the result, you land right back where you started. It proves that every continuous function has an **antiderivative**.

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2. FTC Part 2: The Evaluation Theorem

This is the part you likely use most often in homework. It provides a shortcut to calculate the definite integral without having to use those complicated Riemann sums (rectangles).

The Statement: If f is continuous on $[a,b]$ and F is **any** antiderivative of f (meaning $F'=f$), then:

$\int_a^b f(x)\,dx=F(b)-F(a)$

What this means: To find the area under $f(x)$ from a to b, you don't need to draw rectangles. You just find the antiderivative F, plug in the top number, plug in the bottom number, and subtract.

Example: Calculate $\int_1^3 x^2\,dx$.

1. Find the antiderivative of x^2: $F(x)=\frac{x^3}{3}$.
2. Evaluate at the boundaries: $F(3)-F(1)=\frac{3^3}{3}-\frac{1^3}{3}$.
3. Result: $9-\frac{1}{3}=8.67$.

Why is it "Fundamental"?

Before this theorem, finding the area under a curve was incredibly difficult and required summing infinite series of shapes. The FTC turned a geometry problem (Area) into an algebraic problem (Antiderivatives).

Feature	Differential Calculus	Integral Calculus

Feature	Differential Calculus	Integral Calculus
Focus	Instantaneous rate of change.	Total accumulation / Area.
Tool	The Derivative (f').	The Integral ($\int$).
The Connection	The slope of the area function is the height of the curve.	The area is the "net change" of the antiderivative.

3. The Mean Value Theorem for Integrals

Closely related to the FTC is the idea that for a continuous function on $[a,b]$, there is at least one point c where the height of the function $f(c)$ represents the **average value** of the function.

Mathematically:

$$\int_a^b f(x)dx=f(c)(b-a)$$

This means the area under the curve is exactly equal to the area of a single rectangle with width $(b-a)$ and height $f(c)$.

MEAN VALUE THEOREM OF INTEGRAL CALCULAS

The **Mean Value Theorem for Integrals (MVTI)** is a beautiful concept that bridges the gap between the "squiggly" area under a curve and a simple rectangle.

In plain English: If you have a continuous function on an interval, there is at least one "average" height that, if used to draw a single flat rectangle, would give you the **exact same area** as the area under the curve.

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1. The Formal Statement

If f is continuous on the closed interval $[a,b]$, then there exists at least one number c in the interval (a,b) such that:

$$\int_a^b f(x)dx=f(c)(b-a)$$

Breaking down the formula:

- $\int_a^b f(x)dx$: The **actual area** under the curve from a to b.
- $f(c)$: The "mean" or **average value** of the function.
- $(b-a)$: The **width** of the interval.

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2. The Concept: Average Value of a Function

The MVTI is most commonly used to find the **average value** (f_{avg}) of a function over an interval. If you rearrange the formula, you get:

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$$

Think of it this way: Imagine the area under the curve is made of water in a tank. If you let the water "settle" until it is perfectly flat (but the total volume stays the same), the height of that flat water is $f(c)$.

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3. An Example in Action

Let's find the average value of $f(x)=x^2$ on the interval $[0,3]$.

1. **Calculate the Integral:**

   $$\int_0^3 x^2 dx = \left[\frac{x^3}{3}\right]_0^3 = \frac{27}{3} - 0 = 9$$

2. **Apply the Theorem:** The width $(b-a)$ is $3-0=3$. So, $f(c)\cdot 3 = 9 \Longrightarrow f(c)=3$.
3. **Find c:** Where does $x^2=3 ? x=3$ (which is approx 1.732). Since 1.732 is inside our interval $[0,3]$, the theorem is satisfied!

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4. Why is it important?

- **Physics:** If $f(x)$ represents your velocity, the MVTI tells you that there was at least one moment in time where your **instantaneous speed** was exactly equal to your **average speed** for the whole trip.
- **Proof Building:** This theorem is a vital tool used to prove the **Fundamental Theorem of Calculus**.
- **Estimation:** It allows us to estimate the value of an integral if we know the maximum and minimum values of a function.

Comparison: Derivatives vs. Integrals

There is also a "Mean Value Theorem" for derivatives. Here's how they compare:

Theorem	Focus	Guarantees
MVT for Derivatives	Slope (Rate of change)	There is a point where the **instantaneous slope** equals the **average slope**.
MVT for Integrals	Area (Accumulation)	There is a point where the **instantaneous height** equals the **average height**.

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- $(b-a)$: The **width** of the interval.

2. The Concept: Average Value of a Function

The MVTI is most commonly used to find the **average value** (f_{avg}) of a function over an interval. If you rearrange the formula, you get:

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Think of it this way: Imagine the area under the curve is made of water in a tank. If you let the water "settle" until it is perfectly flat (but the total volume stays the same), the height of that flat water is $f(c)$.

---

3. An Example in Action

Let's find the average value of $f(x)=x^2$ on the interval $[0,3]$.

1. **Calculate the Integral:**

   $$\int_0^3 x^2\,dx = \left[\frac{x^3}{3}\right]_0^3 = \frac{27}{3} - 0 = 9$$

2. **Apply the Theorem:** The width $(b-a)$ is $3-0=3$. So, $f(c)\cdot 3 = 9 \Longrightarrow f(c) = 3$.
3. **Find c:** Where does $x^2=3$?$x=\sqrt{3}$ (which is approx 1.732). Since 1.732 is inside our interval $[0,3]$, the theorem is satisfied!

---

4. Why is it important?

- **Physics:** If $f(x)$ represents your velocity, the MVTI tells you that there was at least one moment in time where your **instantaneous speed** was exactly equal to your **average speed** for the whole trip.
- **Proof Building:** This theorem is a vital tool used to prove the **Fundamental Theorem of Calculus**.
- **Estimation:** It allows us to estimate the value of an integral if we know the maximum and minimum values of a function.

Comparison: Derivatives vs. Integrals

There is also a "Mean Value Theorem" for derivatives. Here’s how they compare:

Theorem	Focus	Guarantees

Theorem	Focus	Guarantees
MVT for Derivatives	Slope (Rate of change)	There is a point where the **instantaneous slope** equals the **average slope**.
MVT for Integrals	Area (Accumulation)	There is a point where the **instantaneous height** equals the **average height**.

UNIT -3

IMPROPER INTEGRALS

An **Improper Integral** is a definite integral that "breaks" at least one of the standard rules for Riemann integration.

In a standard Riemann integral, the interval $[a,b]$ must be **finite** and the function $f(x)$ must be **bounded**. If either of these conditions fails, the integral becomes "improper." To solve them, we use **limits** to see if the area approaches a specific number (converges) or grows without bound (diverges).

---

1. Type 1: Infinite Intervals

These occur when one or both of the limits of integration are infinite. We are calculating the area under a curve as it stretches forever along the x-axis.

- **Format:** $\int_a^\infty f(x)dx$ or $\int_{-\infty}^b f(x)dx$
- **How to solve:** Replace the infinity with a variable (t) and take the limit.

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$$

Example: $\int_1^\infty \frac{1}{x^2}dx$

1. $\lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t$
2. $\lim_{t\to\infty}\left(-\frac{1}{t}+1\right)=0+1=1.$
3. Since the limit is a finite number, we say the integral **converges**.

---

2. Type 2: Discontinuous Integrands

These occur when the interval $[a,b]$ is finite, but the function has a **vertical asymptote** (goes to infinity) somewhere in that interval.

- **At an endpoint:** If $f(x)$ is discontinuous at b:

  $\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx$

- **In the middle:** If $f(x)$ is discontinuous at c (where $a<c<b$), you must split the integral into two pieces:

  $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Example: $\int_0^1 \frac{1}{\sqrt{x}}dx$ (The function is undefined at $x=0$).

1. $\lim_{t\to 0^+}\int_t^1 x^{-1/2}dx$
2. $\lim_{t\to 0^+}[2\sqrt{x}]_t^1 = 2(1)-0=2$. (Converges)

---

3. Convergence vs. Divergence

- **Convergent:** The limit exists and is a finite number. The "infinite" shape actually has a measurable, finite area.
- **Divergent:** The limit does not exist (it goes to ∞ or $-\infty$). The area is truly infinite.

The p-test (A Quick Shortcut)

A very common tool for checking convergence of power functions:

$\int_1^\infty \frac{1}{x^p}dx$

- **Converges** if $p>1$.
- **Diverges** if $p\le 1$.

Note: This leads to "Gabriel's Horn" paradox—a shape with finite volume but infinite surface area!

---

4. Comparison Test

Sometimes an integral is too hard to solve directly. We can compare it to a simpler function:

- **Direct Comparison:** If $0 \le f(x) \le g(x)$, and $\int g(x)$ converges, then the smaller $\int f(x)$ must also converge.
- **Limit Comparison:** If the ratio of two functions $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ is a positive constant, they both behave the same way (both converge or both diverge).

Summary Table

Problem Type	Appearance	Method
Infinite Limit	$\int_1^\infty \dots$	Use $\lim_{t \to \infty}$
Vertical Asymptote	$\int_0^1 \frac{1}{x} \dots$	Use $\lim_{t \to 0^+}$
Both Ends	$\int_{-\infty}^\infty \dots$	Split at 0 and use two limits.

CONVERGENCE OF IMPROPER INTEGRAL

Determining the **convergence** of an improper integral means deciding if the area under the curve settles at a finite number or grows to infinity. We don't always need to solve the integral to know if it converges; we can often use **tests** to predict its behavior.

---

1. Convergence of Type 1 (Infinite Intervals)

For an integral like $\int_a^\infty f(x)dx$, convergence depends on how fast the function "decays" as it moves toward infinity.

- **Convergent:** The function drops to zero fast enough that the accumulated area is finite.
- **Divergent:** The function drops to zero too slowly (or not at all), so the area keeps piling up forever.

The *p*-Series Test (The "Gold Standard")

This is the most common shortcut used in calculus. For $\int_1^\infty \frac{1}{x^p} dx$:

- **Converges** if $p > 1$ (e.g., $1/x^2$, $1/x^{1.1}$).
- **Diverges** if $p \le 1$ (e.g., $1/x$, $1/\sqrt{x}$).

---

2. Convergence of Type 2 (Infinite Discontinuities)

For an integral like $\int_0^1 \frac{1}{x^p} dx$, where the function blows up at $x=0$:

- **Converges** if $p<1$ (e.g., $1/x$).
- **Diverges** if $p\geq 1$ (e.g., $1/x$).

Notice the flip: A function like $1/x^2$ converges at infinity but diverges at zero. A function like $1/x$ diverges at infinity but converges at zero.

---

3. The Comparison Tests

When an integral is too complex to solve (like $\int_1^\infty xe^{-x} dx$), we compare it to a "parent" function we already understand.

A. Direct Comparison Test (DCT)

Suppose $f(x)$ and $g(x)$ are continuous and $0\leq f(x)\leq g(x)$ for all $x\geq a$:

1. **If the "Big" one converges**, the "Small" one must also converge. (If $\int g(x)$ is finite, $\int f(x)$ is finite).
2. **If the "Small" one diverges**, the "Big" one must also diverge. (If $\int f(x)$ is infinite, $\int g(x)$ is infinite).

B. Limit Comparison Test (LCT)

If the functions are messy, we look at their "ultimate" behavior. If $f(x)>0$ and $g(x)>0$, and:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \quad (0<L<\infty)$$

Then $\int f(x)$ and $\int g(x)$ **behave exactly the same way.** They either both converge or both diverge.

---

4. Summary Checklist for Convergence

Test	Best Used When...	Logic

Test	Best Used When...	Logic
Direct Calculation	The antiderivative is easy to find.	If $\lim_{t\to\infty} F(t)$ exists, it converges.
***p*-Test**	You have a simple power of x.	Compare p to 1.
Direct Comparison	You can easily say one function is "smaller" than another.	Large converges $\longrightarrow$ Small converges.
Limit Comparison	You have messy polynomials or roots.	Keep only the highest powers to see the "true" nature.

A Common Example: The "Harmonic" Trap

Does $\int_1^\infty \frac{1}{x} dx$ converge? Even though $\lim_{x\to\infty} \frac{1}{x} = 0$ (the height goes to zero), it doesn't go to zero **fast enough**.

$$\lim_{t\to\infty} [\ln(x)]_1^t = \lim_{t\to\infty} \ln(t) = \infty$$

Therefore, it **diverges**.

COMPARISION TEST

IN mathematical analysis, **Comparison Tests** are the primary tools used to determine the convergence of an improper integral when you cannot (or don't want to) find the actual antiderivative.

They work by comparing a "messy" function to a "benchmark" function—usually a p-series ($1/x^p$) or an exponential (e^x)—whose behavior is already known.

---

1. Direct Comparison Test (DCT)

This is the most intuitive test. It relies on the "Squeeze" logic. Suppose $f(x)$ and $g(x)$ are continuous functions where $0 \le f(x) \le g(x)$ for all x in the interval.

- **Case A: If the "Big" function converges, the "Small" one must converge.** If $\int_a^\infty g(x)dx$ is finite, then $\int_a^\infty f(x)dx$ must also be finite.
- **Case B: If the "Small" function diverges, the "Big" one must diverge.** If $\int_a^\infty f(x)dx = \infty$, then $\int_a^\infty g(x)dx$ must also be ∞.

Example: Consider $\int_1^\infty \frac{1}{x^3+5}dx$. We know that $x^3+5>x^3$. Therefore, $\frac{1}{x^3+5}<\frac{1}{x^3}$. Since $\int_1^\infty \frac{1}{x^3}dx$ converges ($p=3>1$), our original integral **converges**.

---

2. Limit Comparison Test (LCT)

Sometimes the Direct Comparison Test is hard to use because the inequalities don't work out nicely (e.g., if the denominator has a minus sign). The **Limit Comparison Test** looks at the "ultimate" relationship between two functions as x goes to infinity.

If $f(x)>0$ and $g(x)>0$, we calculate the limit:

$$L=\lim_{x\to\infty}\frac{f(x)}{g(x)}$$

- **If L is a finite, positive number ($0<L<\infty$):** Both $\int f(x)$ and $\int g(x)$ behave exactly the same way. If one converges, both converge. If one diverges, both diverge.
- **If $L=0$:** $f(x)$ is "smaller" than $g(x)$ in the long run. If $\int g(x)$ converges, then $\int f(x)$ converges.
- **If $L=\infty$:** $f(x)$ is "larger" than $g(x)$ in the long run. If $\int g(x)$ diverges, then $\int f(x)$ diverges.

[Image showing the Limit Comparison Test logic for f(x) and g(x)]

---

3. How to Pick a Comparison Function ($g(x)$)

Choosing the right $g(x)$ is the secret to success. Use these "dominant term" rules:

1. **Polynomials:** Keep only the highest power of x.
   - *Example:* For $f(x)=\frac{x}{x^3-5x+2}$, use $g(x)=\frac{x}{x^3}=\frac{1}{x^2}$.
2. **Roots:** Keep the highest power inside the root.
   - *Example:* For $f(x)=\frac{1}{\sqrt{x^4+1}}$, use $g(x)=\frac{1}{\sqrt{x^4}}=\frac{1}{x^2}$.
3. **Exponentials:** Compare to e^x.
   - *Example:* For $f(x)=\frac{1}{e^x+x}$, use $g(x)=\frac{1}{e^x}$.

Summary Checklist

If you think it...	Use this Logic	Result
Converges	Find a **larger** function that converges.	$f \le g$ and $\int g < \infty \Longrightarrow \int f < \infty$
Diverges	Find a **smaller** function that diverges.	$f \ge g$ and $\int g = \infty \Longrightarrow \int f = \infty$
Is "Messy"	Use Limit Comparison.	$f \approx g \Longrightarrow$ same behavior

ABELS AND DRICHLET TEST

When the Comparison Test isn't enough—usually because the function alternates between positive and negative values—we turn to **Abel's Test** and **Dirichlet's Test**.

These tests are specifically designed for integrals of the form:

$\int_a^\infty f(x)g(x)dx$

They are the integral versions of the tests used for infinite series.

---

1. Dirichlet's Test

Dirichlet's Test is a powerful way to prove convergence for functions that oscillate (like $\sin(x)$ or $\cos(x)$) but are dampened by another function.

Criteria: An integral $\int_a^\infty f(x)g(x)dx$ converges if:

1. **Bounded Integral:** The integral of $f(x)$ is bounded for all $t > a$. That is, $|\int_a^t f(x)dx| \le K$ for some constant K.
2. **Monotonicity:** $g(x)$ is a monotonic function (always decreasing or always increasing).
3. **Limit to Zero:** $\lim_{x \to \infty} g(x) = 0$.

Classic Example: $\int_0^\infty \frac{\sin(x)}{x}dx$

- $f(x) = \sin(x)$ has a bounded integral (it just bounces between 0 and 2).
- $g(x) = \frac{1}{x}$ is monotonic and goes to 0 as $x \to \infty$.
- **Result:** The integral converges.

---

2. Abel's Test

Abel's Test is used when you already know that one part of the integral converges, and you want to see if multiplying it by another "well-behaved" function keeps it convergent.

Criteria: An integral $\int_a^\infty f(x)g(x)dx$ converges if:

1. **Known Convergence:** $\int_a^\infty f(x)dx$ is already known to be convergent.
2. **Boundedness:** $g(x)$ is bounded on $[a,\infty)$.
3. **Monotonicity:** $g(x)$ is monotonic.

Key Difference: Unlike Dirichlet's, $g(x)$ in Abel's test does **not** have to go to zero; it just needs to be monotonic and stay within a certain range (bounded).

Summary Comparison

Feature	Dirichlet's Test	Abel's Test
Requirement for $\int f(x)$	Must be **bounded** (not necessarily convergent).	Must be **convergent**.
Requirement for $g(x)$	Monotonic AND $\to 0$.	Monotonic AND **bounded**.
Common Use Case	Oscillating functions like $\frac{\sin x}{x^p}$.	Refining an already convergent integral.

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3. Why These Matter: Absolute vs. Conditional Convergence

These tests are essential for identifying **Conditional Convergence**.

- **Absolute Convergence:** $\int |f(x)|dx$ converges. (The "heavyweight" convergence).
- **Conditional Convergence:** $\int f(x)dx$ converges, but $\int |f(x)|dx$ **diverges**.

The integral $\int_1^\infty \frac{\sin x}{x}dx$ is the most famous example. It converges by Dirichlet's Test, but if you take the absolute value $|\frac{\sin x}{x}|$, the area becomes infinite (similar to the harmonic series $1/x$).

FRULLANI INTEGRAL

A **Frullani Integral** is a specific type of improper integral that allows you to find a definite value for a seemingly complex expression without actually finding the antiderivative. It deals with the difference of the same function scaled by two different constants.

1. The General Formula

The standard form of a Frullani Integral is:

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}\,dx$$

Where $a,b>0$.

The Result:

If the function $f(x)$ is continuous for $x\geq 0$ and the limit as $x\to\infty$ exists, the integral evaluates to:

$$(f(\infty)-f(0))\ln\left(\frac{b}{a}\right)$$

- $\boldsymbol{f(0)}$: The value of the function as $x\to 0$.
- $\boldsymbol{f(\infty)}$: The limit of the function as $x\to\infty$ (denoted as L).

2. A Classic Example: Exponential Decay

Consider the integral:

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx$$

1. **Identify $\boldsymbol{f(x)}$**: Here, $f(x)=e^{-x}$.
2. **Evaluate limits**:
   - $f(0)=e^0=1$
   - $f(\infty)=\lim_{x\to\infty} e^{-x}=0$
3. **Apply formula**:

   $$(0-1)\ln\left(\frac{b}{a}\right)=-\ln\left(\frac{b}{a}\right)=\ln\left(\boldsymbol{\frac{a}{b}}\right)$$

3. Necessary Conditions

For the Frullani formula to work, the following must hold:

- $a,b>0$.
- The integral must be convergent.
- $f(x)$ must be such that the integral $\int_1^\infty \frac{f(t)}{t} dt$ doesn't necessarily converge, but the *difference* in the numerator ensures convergence at infinity and at zero.

4. Why does it work? (The Logic)

The proof usually involves a substitution and changing the order of integration (Fubini's Theorem) or treating it as a double integral:

$$\int_0^\infty \frac{f(ax)-f(bx)}{x} dx = \int_0^\infty \int_b^a f'(yx)\, dy\, dx$$

By switching the order of integration, you integrate with respect to x first, which simplifies the expression into a logarithmic form.

Summary Table

Function Type	$f(0)$	$f(\infty)$	Result
Exponential (e^{-x})	1	0	$\ln(b/a)$
Arctan ($\arctan x$)	0	$\pi/2$	$(\pi/2)\ln(a/b)$
General Case	$f(0)$	L	$(L-f(0))\ln(a/b)$

INTEGRAL AS A FUNCTION OF A PARAMETER

In calculus, treating an integral as a function of a parameter—often called **Leibniz's Rule** or "differentiation under the integral sign"—is a powerful technique for solving complex integrals that seem impossible using standard methods.

Essentially, if you have an integral where the integrand or the limits depend on a variable (let's call it α), the result of that integral is also a function of α.

---

1. The General Form

Consider a function $F(\alpha)$ defined by an integral:

$$F(\alpha)=\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\,dx$$

Here, x is the variable of integration, and α is the **parameter**. To find the derivative of this function with respect to the parameter, we use the **Leibniz Integral Rule**:

$$\frac{d}{d\alpha}F(\alpha)=\int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx + f(b(\alpha),\alpha)\cdot\frac{d}{d\alpha}b - f(a(\alpha),\alpha)\cdot\frac{d}{d\alpha}a$$

Breaking it down:

- **The Integral Term:** You differentiate the "inside" function with respect to the parameter.
- **The Boundary Terms:** These account for how the limits of integration change as α changes (using the Chain Rule).

---

2. Constant Limits (The "Feynman" Technique)

If the limits of integration a and b are constants and do not depend on α, the formula simplifies beautifully:

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx=\int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx$$

This is the version Richard Feynman famously used to crack tough integrals. By introducing a parameter where one doesn't exist, you can turn a hard integration problem into an easier differentiation problem.

---

3. Example: Solving $\int_0^\infty \frac{\sin(x)}{x}\,dx$

This is the famous Dirichlet integral. It's hard to solve directly, so we define a function with a parameter α:

1. **Define:** $I(\alpha)=\int_0^\infty e^{-\alpha x} \frac{\sin(x)}{x} dx$
2. **Differentiate:** $\frac{dI}{d\alpha}=\int_0^\infty \frac{\partial}{\partial\alpha}\left(e^{-\alpha x}\frac{\sin(x)}{x}\right)dx=\int_0^\infty -e^{-\alpha x}\sin(x)dx$
3. **Integrate the Result:** This is a standard integral: $\frac{dI}{d\alpha}=-\frac{1}{\alpha^2+1}$
4. **Anti-differentiate:** $I(\alpha)=-\arctan(\alpha)+C$
5. **Solve for C:** As $\alpha\to\infty$, $I(\alpha)\to 0$, so $C=\pi/2$.
6. **Final Answer:** Set $\alpha=0$ to get the original integral: $I(0)=\pi/2$.

Key Requirements for Use

For this technique to be valid, the following conditions must usually be met (based on the **Leibniz Theorem**):

- The function $f(x,\alpha)$ and its partial derivative $\frac{\partial f}{\partial\alpha}$ must be **continuous** in the region of integration.
- If the integral is improper (limits are infinity), the integral must **converge uniformly**.

CONTINUITY,DIFFERENTIABILTY AND INTEGRABILITY OF AN INTEGRAL OF A FUNCTION OF A PARAMETER

When we define a function using an integral, such as $F(\alpha)=\int_a^b f(x,\alpha)dx$, we aren't just calculating a number; we are creating a new mathematical object. To use $F(\alpha)$ in calculus, we need to know if it behaves "nicely"—meaning, is it continuous, can we differentiate it, and can we integrate it again?

These properties generally depend on the behavior of the "inside" function, $f(x,\alpha)$.

---

1. Continuity of the Integral

If you nudge the parameter α slightly, does the value of the integral change slightly?

- **The Condition:** If $f(x,\alpha)$ is **continuous** on the rectangle $[a,b]\times[c,d]$, then the function $F(\alpha)$ is continuous for all $\alpha\in[c,d]$.
- **Intuition:** If the surface $z=f(x,\alpha)$ has no rips or jumps, the area under the curve for a fixed α will vary smoothly as you slide α along its axis.

2. Differentiability (Leibniz's Rule)

This is the most "famous" property. It tells us when we can move the derivative operator inside the integral sign.

- **The Condition:** For $F'(\alpha)$ to exist:
  1. $f(x,\alpha)$ must be continuous.
  2. The partial derivative $\frac{\partial f}{\partial \alpha}$ must exist and be continuous.
- **The Formula:**

  $$\frac{d}{d\alpha}\int_a^b f(x,\alpha)dx=\int_a^b \frac{\partial}{\partial \alpha} f(x,\alpha)dx$$

- **The Logic:** This essentially treats the integral as a giant sum. Since the derivative of a sum is the sum of the derivatives, this works as long as the functions are smooth enough to allow the limit processes to swap places.

3. Integrability (Changing the Order)

Can we integrate the result of an integral? This leads to **Fubini's Theorem** for double integrals.

- **The Condition:** If $f(x,\alpha)$ is continuous on the region, you can integrate $F(\alpha)$ with respect to α.
- **The Rule:** You can switch the order of integration:

  $$\int_c^d\left(\int_a^b f(x,\alpha)dx\right)d\alpha=\int_a^b\left(\int_c^d f(x,\alpha)d\alpha\right)dx$$

- **Why use it?** Sometimes $\int f(x,\alpha)dx$ is impossible to solve, but $\int f(x,\alpha)d\alpha$ is easy. Switching the order can turn a "brick wall" problem into a simple one.

Summary Table: The "Requirements"

Property	Requirement for $f(x,\alpha)$	Result for $F(\alpha)$
Continuity	f is continuous	$F(\alpha)$ is continuous
Differentiability	$\frac{\partial f}{\partial \alpha}$ is continuous	$F'(\alpha)=\int \frac{\partial f}{\partial \alpha}dx$
Integrability	f is continuous	$\int F(\alpha)d\alpha$ exists (Order can be swapped)

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Important Note: Improper Integrals

If the limits are infinite (e.g., $\int_0^\infty$), these rules become much stricter. You cannot simply swap derivatives or integrals unless the integral **converges uniformly**. If it doesn't converge uniformly, you might get a mathematically "correct" derivation that leads to a

EXAMPLE

To see how these properties work in practice, let’s look at a single function $f(x,\alpha)$ and test it against all three criteria.

Consider the function defined by the integral:

$$F(\alpha)=\int_0^1 \alpha x^{\alpha-1} dx \text{ for } \alpha>0$$

---

1. Continuity of $F(\alpha)$

To check if $F(\alpha)$ is continuous, we first evaluate the integral directly:

$$F(\alpha)=[x^\alpha]_0^1=1^\alpha-0^\alpha=1$$

Since $F(\alpha)=1$ (a constant) for all $\alpha>0$, the function is **continuous**.

The Rule: Because $f(x,\alpha)=\alpha x^{\alpha-1}$ is continuous on the interval $x\in[0,1]$ for any fixed $\alpha>0$, the resulting integral $F(\alpha)$ must be continuous.

---

2. Differentiability (Leibniz’s Rule)

Suppose we want to find $F'(\alpha)$. Since $F(\alpha)=1$, we know the derivative should be **0**. Let’s see if we can get that by differentiating *inside* the integral:

$$\frac{d}{d\alpha}\int_0^1 \alpha x^{\alpha-1} dx \overset{?}{=} \int_0^1 \frac{\partial}{\partial\alpha}(\alpha x^{\alpha-1}) dx$$

The Partial Derivative:

$$\frac{\partial}{\partial\alpha}(\alpha x^{\alpha-1})=x^{\alpha-1}+\alpha x^{\alpha-1}\ln(x)$$

The Result: If we integrate this from 0 to 1, we indeed get 0. However, notice that if $\alpha\le1$, the $\ln(x)$ term creates a singularity at $x=0$.

- **Verdict:** $F(\alpha)$ is differentiable, but we must be careful. If the partial derivative $\frac{\partial f}{\partial \alpha}$ is not continuous or the integral doesn't converge uniformly, Leibniz's Rule might fail.

---

3. Integrability (Changing the Order)

Let's try to calculate the integral of $F(\alpha)$ over the range $\alpha \in [1,2]$:

$$\int_1^2 F(\alpha)\,d\alpha = \int_1^2 \left(\int_0^1 \alpha x^{\alpha-1}\,dx\right) d\alpha$$

Method A (Direct): We already know $F(\alpha)=1$. So, $\int_1^2 1\,d\alpha = \mathbf{1}$.

Method B (Switching Order):

$$\int_0^1 \left(\int_1^2 \alpha x^{\alpha-1}\,d\alpha\right) dx$$

The inner integral $\int_1^2 \alpha x^{\alpha-1}\,d\alpha$ can be solved using integration by parts or recognizing it as a specific form. After evaluating the inner and then the outer integral, you will again arrive at **1**.

The Rule: Since $f(x,\alpha)$ is continuous on the square $[0,1]\times[1,2]$, Fubini's Theorem guarantees that we can switch the order of integration and get the same result.

Summary Checklist for a Question

If you are asked to "Examine the continuity/differentiability/integrability" of an integral $F(\alpha)$:

1. **Continuity:** Is $f(x,\alpha)$ continuous? If yes, $F(\alpha)$ is continuous.
2. **Differentiability:** Is $\frac{\partial f}{\partial \alpha}$ continuous? If yes, you can use Leibniz's Rule.
3. **Integrability:** Is $f(x,\alpha)$ bounded and continuous? If yes, you can swap the order of $\int dx$ and $\int d\alpha$.

UNIT 4

JACOBIANS

Think of a **Jacobian** as a "scaling factor" for multivariable transformations. Just as the derivative $f'(x)$ tells you how much a function stretches or shrinks a single line, the Jacobian tells you how much a transformation stretches or shrinks **area** (in 2D) or **volume** (in 3D).

It is essentially the multivariable version of the "substitution rule" ($du=g'(x)dx$) from basic calculus.

---

1. The Jacobian Matrix

If you have a set of functions that transform coordinates (u,v) into (x,y):

- $x=g(u,v)$
- $y=h(u,v)$

The **Jacobian Matrix** J is the matrix of all first-order partial derivatives:

$$J=\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

---

2. The Jacobian Determinant

When we talk about "The Jacobian" in the context of integration, we usually mean the **determinant** of that matrix, denoted as $\frac{\partial(x,y)}{\partial(u,v)}$ or $|J|$:

$$\det(J)=\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Why the determinant? The determinant of a matrix represents the area of the parallelogram formed by its vector columns. In calculus, this tells us how a tiny rectangle $dudv$ in the uv-plane transforms into a "warped" shape in the xy-plane.

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3. Application: Change of Variables in Integrals

When you change variables in a double integral, you must multiply by the **absolute value** of the Jacobian to keep the "mass" of the integral correct:

$$\iint_R f(x,y)\,dx\,dy=\iint_S f(x(u,v),y(u,v))\cdot\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$

Classic Example: Polar Coordinates

To switch from Cartesian (x,y) to Polar (r,θ):

- $x=r\cos\theta$
- $y=r\sin\theta$

The Jacobian matrix is:

$$J=\begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix}$$

The determinant is:

$$\det(J)=(\cos\theta)(r\cos\theta)-(-r\sin\theta)(\sin\theta)=r\cos^2\theta+r\sin^2\theta=\mathbf{r}$$

This is exactly why $dx\,dy$ becomes $\mathbf{r}\,dr\,d\theta$. The "r" is the Jacobian!

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4. Properties of Jacobians

- **The Inverse Property:** If you know the Jacobian for $(u,v)\to(x,y)$, the inverse transformation $(x,y)\to(u,v)$ is simply the reciprocal:

  $$\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$$

- **Chain Rule:** If you transform from $(u,v)\to(r,s)\to(x,y)$, the Jacobians multiply just like derivatives:

  $$\frac{\partial(u,v)}{\partial(x,y)}=\frac{\partial(r,s)}{\partial(x,y)}\cdot\frac{\partial(u,v)}{\partial(r,s)}$$

Summary Checklist

Feature	Meaning
Matrix	The collection of all local slopes (partial derivatives).

Feature	Meaning
Determinant	The local expansion/compression factor of space.
If $J=0$	The transformation is "degenerate" (it collapses a 2D area into a 1D line).
Usage	Essential for changing variables in multiple integrals.

BETA AND GAMMA FUNCTION

The **Beta** and **Gamma** functions are known as "Special Functions." They are essentially extensions of the factorial concept into the world of continuous numbers and complex variables. They are deeply linked to each other and are indispensable for solving definite integrals that don't have elementary antiderivatives.

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1. The Gamma Function $\Gamma(n)$

The Gamma function is the continuous version of the factorial. While $n!$ is only defined for integers, $\Gamma(n)$ works for almost all real and complex numbers.

The Definition: For $n>0$:

$$\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}dx$$

Key Properties:

- **The Factorial Connection:** $\Gamma(n+1)=n\Gamma(n)$. For an integer n, $\Gamma(n)=(n-1)!$.
- **Specific Value:** $\Gamma(1)=1$.
- **The "Half" Value:** $\Gamma(1/2)=\pi$. (This is a famous result derived from Gaussian integrals).

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2. The Beta Function $B(m,n)$

The Beta function is a "binary" function (it takes two parameters) and is defined over a finite interval, usually $[0,1]$.

The Definition: For $m,n>0$:

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$$

Trigonometric Form: Sometimes it's easier to solve integrals using sines and cosines:

$$B(m,n)=2\int_0^{\pi/2}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}d\theta$$

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3. The Bridge: Relationship Between Beta and Gamma

The most important identity in this topic is the one that links these two functions together. It allows you to solve a Beta integral by breaking it into Gamma parts:

$$B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$B(5,6)=\frac{4!\cdot 5!}{10!}=\frac{24\cdot 120}{3,628,800}=\frac{1}{1260}$$

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4. Summary Table of Identities

Property	Formula
Recurrence	$\Gamma(n+1)=n\Gamma(n)$
Symmetry	$B(m,n)=B(n,m)$
Reflection	$\Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin(n\pi)}$
Duplication	$\Gamma(n)\Gamma(n+1/2)=2^{1-2n}\sqrt{\pi}\,\Gamma(2n)$

A Quick "Cheat" for Integrals

If you see an integral in the form $\int_0^{\pi/2}\sin^p\theta\cos^q\theta\, d\theta$, the answer is always:

$$\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

DOUBLE AND TRIPLE INTEGRALS

In calculus, moving from single integrals to double and triple integrals is like moving from calculating the **area under a curve** to calculating the **volume of a solid** or the **mass of an object** with varying density.

1. Double Integrals ($\iint_R f(x,y)dA$)

A double integral calculates the volume under a surface $z=f(x,y)$ over a region R in the xy-plane.

- **Rectangular Coordinates:** If the region is a rectangle $[a,b]\times[c,d]$, we use **Fubini's Theorem** to integrate one variable at a time:

  $$\iint_R f(x,y)dxdy=\int_c^d\left(\int_a^b f(x,y)dx\right)dy$$

- **General Regions:** If the boundaries are curves (e.g., y goes from $g_1(x)$ to $g_2(x)$), you must set up the limits carefully. The "outer" integral must always have constant limits.

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2. Triple Integrals ($\iiint_V f(x,y,z)dV$)

Triple integrals are used to find the "total" of something within a 3D volume V. If $f(x,y,z)=1$, the integral simply gives the **Volume** of the shape. If f represents density, the integral gives the **Mass**.

- **Order of Integration:** There are 6 possible orders ($dzdydx$, $dxdzdy$, etc.). Choosing the right one depends on which boundary is easiest to describe as a function of the others.

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3. Change of Variables (The "Big Three" Systems)

Often, calculating these in standard (x,y,z) coordinates is a nightmare. We use the **Jacobian** to switch to systems that match the symmetry of the shape:

A. Polar Coordinates (2D)

Used for circles, pies, or cylinders.

- **Substitution:** $x=r\cos\theta$, $y=r\sin\theta$
- **Differential:** $dA=\mathbf{r}\,dr\,d\theta$

B. Cylindrical Coordinates (3D)

Best for pipes, cans, or objects symmetric around the z-axis. It's just Polar + z.

- **Differential:** $dV = \mathbf{r}\, dz\, dr\, d\theta$

C. Spherical Coordinates (3D)

Best for spheres, cones, or globes.

- **Substitution:** $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$
- **Differential:** $dV = \boldsymbol{\rho^2 \sin\phi}\, d\rho\, d\phi\, d\theta$

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4. Applications of Multiple Integrals

Beyond just volume, these are the workhorses of physics and engineering:

- **Mass:** $M = \iiint_V \rho(x,y,z)\, dV$
- **Center of Mass:** $(\bar{x}, \bar{y}, \bar{z})$ where each coordinate is the "moment" divided by total mass.
- **Moment of Inertia:** How hard it is to spin an object around an axis.
- **Average Value:** The average "height" or "temperature" over a region.

Comparison Summary Table

Feature	Double Integral	Triple Integral
Domain	Region R in 2D plane	Solid V in 3D space
Common Use	Area, Volume under surface	Volume, Mass, Center of Gravity
Diff. Element (dA/dV)	$dx\,dy$ or $r\,dr\,d\theta$	$dx\,dy\,dz$ or $\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

DRICHLET S INTEGRAL

Dirichlet's Integral typically refers to a specific, famous improper integral that evaluates to a surprisingly clean constant. In a broader context, it can also refer to **Dirichlet's**

Multiple Integral Formula, which is a powerful tool for solving integrals over n-dimensional regions.

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1. The Classic Dirichlet Integral

The most famous "Dirichlet Integral" is the integral of the sinc function over a semi-infinite interval:

$$\int_{0}^{\infty} x\sin(x)dx=2\pi$$

Why is it special?

- **Non-Elementary:** The function $f(x)=x\sin(x)$ does not have an antiderivative that can be expressed in terms of basic functions (like polynomials or logs).
- **Conditional Convergence:** The integral converges, but it is not *absolutely* convergent. This means the area "above" and "below" the x-axis cancels out in a very specific way to reach $\pi/2$.

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2. Dirichlet's Multiple Integral Formula

This is a generalization used to evaluate integrals over a specific n-dimensional region (a simplex). It is essentially a shortcut that uses the **Gamma Function**.

If the region V is defined by $x\geq 0, y\geq 0, z\geq 0$ and $x+y+z\leq 1$, then:

$$\iiint_V x^{l-1}y^{m-1}z^{n-1}dxdydz=\Gamma(l+m+n+1)\Gamma(l)\Gamma(m)\Gamma(n)$$

Why use it?

It transforms a complex triple (or n-tuple) integral into a simple calculation using factorials/Gamma values. It is particularly useful for finding the **mass or center of gravity** of objects where the density varies as a power of the coordinates.

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3. Liouville's Extension

Dirichlet's formula was extended by Liouville to handle more general cases where the sum of variables isn't just ≤ 1, but falls within a range $[h_1, h_2]$:

$$\iiint_V f(x+y+z) x^{l-1} y^{m-1} z^{n-1} dx\,dy\,dz = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)} \int_{h_1}^{h_2} f(u) u^{l+m+n-1} du$$

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4. Comparison Table: Dirichlet vs. Standard Methods

Feature	Standard Integration	Dirichlet Formula
Effort	Requires setting up 3+ nested limits.	Direct substitution into Gamma formula.
Region	Any shape (often complex).	Specific simplex ($x+y+z \le 1$).
Function Type	Any $f(x,y,z)$.	Power functions ($x^{l-1} y^{m-1} z^{n-1}$).

Example: Volume of a Tetrahedron

To find the volume of the region bounded by $x+y+z \le 1$ and the coordinate planes:

1. Set $l=1, m=1, n=1$ (so the integrand is $x^0 y^0 z^0 = 1$).
2. Apply formula: $\frac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1+1)} = \frac{1 \cdot 1 \cdot 1}{\Gamma(4)}$.
3. Since $\Gamma(4) = 3! = 6$, the volume is **1/6**.

CHANGE OF ORDER OF INTEGRATION IN DOUBLE INTEGRALS

Chang the order of integration is a tactical move in calculus. Often, a double integral is impossible to solve in the order $dydx$, but becomes trivial in the order $dxdy$.

To switch the order, you can't just swap the dx and dy; you must **re-describe the region of integration**.

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1. The Strategy: "Slice and Slide"

When you integrate, you are essentially "scanning" a 2D region.

- **Order $dydx$:** You scan with **vertical sticks**. You slide a vertical line from $x=a$ to $x=b$. The height of the stick is determined by the functions $y=g_1(x)$ (bottom) and $y=g_2(x)$ (top).
- **Order $dxdy$:** You scan with **horizontal sticks**. You slide a horizontal line from $y=c$ to $y=d$. The length of the stick is determined by $x=h_1(y)$ (left) and $x=h_2(y)$ (right).

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2. The Three-Step Process

Step 1: Sketch the Region

Look at the given limits. If you have $\int_0^1\int_x^1 f(x,y)dydx$:

- The outer limits (0 to 1) tell you the x-range.
- The inner limits (x to 1) tell you that y starts at the line $y=x$ and ends at the line $y=1$.
- **The Shape:** A triangle with vertices $(0,0),(0,1)$, and $(1,1)$.

Step 2: Inverse the Functions

To switch to dx, you need x as a function of y.

- If the old boundary was $y=x$, the new boundary is $x=y$.
- If the old boundary was $y=1$ (horizontal), it becomes an endpoint for the y-scan.

Step 3: Write the New Limits

- **New Outer (y):** Look at your sketch. What is the lowest and highest y value in the region? (In our example: 0 to 1).
- **New Inner (x):** For a fixed y, where does a horizontal line enter and exit the shape? (Enters at $x=0$, exits at the line $x=y$).
- **New Integral:** $\int_0^1\int_0^y f(x,y)dxdy$.

---

3. Example: Why we do this

Consider the integral:

$\int_0^1\int_x^1 e^{y^2}dydx$

You **cannot** integrate e^{y^2} with respect to y (there is no elementary antiderivative).

After Switching Order: The region is the same triangle mentioned above. Switching to $dxdy$ gives:

$$\int_0^1 \int_0^y e^{y^2} dx dy$$

1. **Inner Integral:** $\int_0^y e^{y^2} dx = [xe^{y^2}]_0^y = ye^{y^2}$.
2. **Outer Integral:** $\int_0^1 ye^{y^2} dy$.
    - *This is easy!* Use u-substitution ($u = y^2$).
    - Result: $\frac{1}{2}(e-1)$.

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4. When is it "Mandatory"?

1. **Impossible Integrands:** Like e^{y^2}, $\frac{\sin y}{y}$, or $1+x^3$ where the inner variable has no antiderivative.
2. **Boundary Complexity:** Sometimes the region is split into two parts in one orientation but is a single piece in another.